$M/G/1 + D$ Perishable Inventory Systems

Onno Boxma*†, Offer Kella‡§, David Perry¶‖ and Wolfgang Stadje**

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Abstract

We consider a Perishable Inventory System (PIS) in which demands for items arrive according to a Poisson process and items according to a renewal process. Stored items have a deterministic maximum lifetime 'on the shelf'. Exploiting a relation between the so-called Virtual Outdating Time (VOT) process of this PIS and the workload process of the $M/G/1 + D$ queue, we prove a decomposition property of each of these two processes.

Subsequently we analyze two generalizations of the above PIS, where the quality of items on the shelf is not constant. In the first one there are two types of items, with different maximum lifetimes. In the second, the quality of an item gradually deteriorates with age.

1 Introduction

This paper is devoted to the study of a classical Perishable Inventory System (PIS) and two generalizations. The theory of PIS deals with the following phenomenon. Items of a certain type arrive at a collecting point from where they are removed by incoming demands. If an item stays too long 'on the shelf' it becomes outdated (useless by regulation), due to random deterioration or a predetermined expiration time. The standard example is a storage space for commodities, but other applications include blood banks, spot markets for special goods and distribution sites for transplantation organs. An early paper on

*EURANDOM and Department of Mathematics and Computer Science, Eindhoven University of Technology, the Netherlands. o.j.boxma@tue.nl

†Partly funded by the NWO Gravitation project Networks, grant number 024.002.003.

‡Department of Statistics, The Hebrew University of Jerusalem; Jerusalem 9190501, Israel. offer.kella@gmail.com

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¶Industrial Engineering and Technology Management, Holon Institute of Technology, Holon 5810201, Israel. davidper@hit.ac.il

‖Partially supported by ISF grant 3336/24.

**Institute of Mathematics, University of Osnabrück, 49076 Osnabrück, Germany. wstadje@uos.de

the topic is [19], where the arrivals of both demands and items follow Poisson processes. Many references can be found in the surveys of Karaesmen et al. [18] and Krishnamoorthy et al. [21]. The recent survey [12] provides a unifying presentation of the literature, as well as several new model variants. The unifying aspect is to focus on the oldest item on the shelf, and to relate its age to the workload in a particular queueing model. This is also the starting point in the present paper.

Throughout this paper we assume that items have a fixed maximum lifetime on the shelf, which is without loss of generality set equal to one, and items are assigned to demands according to the First-In-First-Out (FIFO) discipline. Now let $A(t)$ denote the age of the oldest item on the shelf at time t or, if the shelf is empty at t, let it denote a 'negative age', defined to be minus the time until the next arrival at the shelf. Furthermore, let $V(t) := 1 - A(t)$; $V(t)$ is the time that would pass from t onwards until the next outdating if no new demands arrive in time, i.e., the remaining lifetime of the oldest item. The process $\mathbf{V}$ $:= \{V(t), t \geq 0\}$ is called the Virtual Outdating Time process (VOT), and plays a key role in our analysis of PIS. V denotes a random variable having the steady state distribution of the VOT process. It is important to realize that the VOT process can be interpreted as the workload process of a specific single server queueing model, in which the idle periods are removed and the busy periods glued together; see Figure 1.
The jumps upward of the VOT, from the remaining lifetime value of the oldest item to that of the subsequent oldest item, occur at the epochs of demand arrivals that find at least one item on the shelf (VOT does not exceed one) or outdating epochs, and the jump sizes correspond to interarrival times of items. In the queueing interpretation, demand arrivals become customer arrivals and jump sizes become service times. Furthermore, the finite shelf life results in customers having a patience of length D (without loss of generality, we take $D = 1$); they do not enter the queueing system if their waiting time (excluding service) would exceed one (corresponding to an unsatisfied demand). Knowledge of $\mathbf{V}$ can be used to determine key performance measures like the number of items on the shelf, the outdating rate and the rate of unsatisfied demands [12].

If demand arrivals follow a Poisson process but item arrivals a renewal process, then the VOT process becomes the workload process of an $M/G/1$ queue in which customers have a fixed patience w.r.t. their waiting time (of one time unit) – except that the idle periods of this queue are deleted, cf. outdating point A_6 in the figure. This VOT process is a Markov regenerative process, where the regeneration cycles are the time intervals between successive outdatings (busy periods in the queueing interpretation [12]). The corresponding PIS, to be called $M/G/1 + D$ PIS, has been studied in [20]. However, we present a novel approach that, in particular, reveals an intriguing decomposition property of both the workload in the $M/G/1+D$ queue and the VOT in the $M/G/1+D$ PIS.

In reality, the quality of an item on the shelf might not be constant. It may depend on its supplier (e.g., blood of young vs. old donors); hence we study a generalization of the above-described $M/G/1+D$ PIS in which the items which

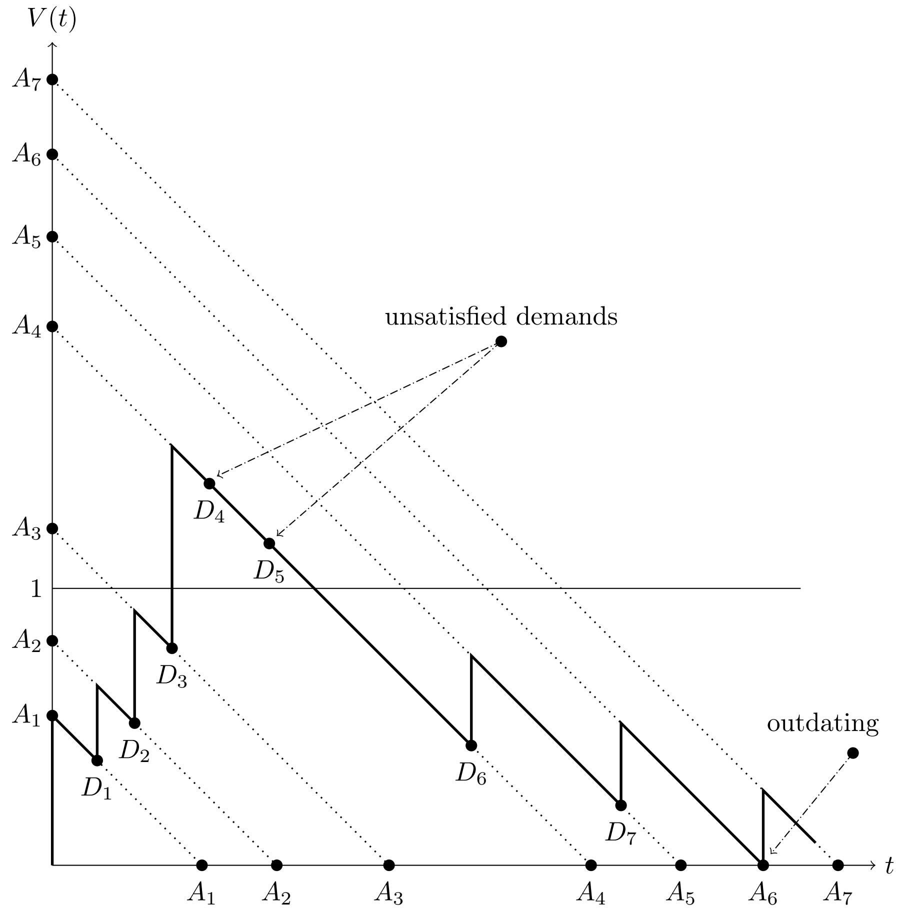

Figure 1: A typical sample path during a single regenerative cycle (between outdatings) of the VOT process. A_n and D_n denote item virtual expiration and demand arrival epochs, respectively.

arrive on the shelf are of two types. One type has expiration time 1 as before, but the other type (say, items from old donors) has a shorter expiration time.

The quality of an item (e.g., a blood portion) may also gradually deteriorate with age. We consider a deterministic function $Q(\cdot)$ that relates the remaining lifetime $V(t)$ of an item to its quality $Q(V(t))$. Under certain assumptions we derive an integral equation for the density of the steady state quality measure $Q(V)$ and we obtain its solution via a translation from the steady state density of V.

Motivation. As indicated above, PIS have a wide range of applicability. In this paper we often refer to blood banks, as they form a natural and important application area. Throughout the paper (in examples in which the rather abstract density of V, often in the form of an infinite sum of convolutions or a

Laplace transform, is made more explicit) we also consider a particular choice of item interarrival distribution that arises naturally in a distribution center that sends a certain item type to two store locations intermittently, where the arrival of these items to the center is according to a Poisson process. This gives rise to Erlang-2 distributed interarrival times at each of the store locations. If one store is much bigger than the other, then a Markovian transfer policy could be more natural. Consider a 2-state Markov process with transition probabilities $p_{11} = 1 - p_{12} = p_1, p_{22} = 1 - p_{21} = p_2 > 0$ and $\exp(\delta)$ sojourn times in each of the two states. Then the intervals between successive visits to state (hospital) 1 have distribution

$$G_1(t) = 1 - \left(1 - \frac{1-p_1}{p_2}\right) \mathrm{e}^{-\delta t} - \frac{1-p_1}{p_2} \mathrm{e}^{-(1-p_2)\delta t}, \quad t \geq 0, \tag{1}$$

and symmetrically for state (hospital) 2. This Coxian distribution is hyperexponential when $1 - p_2 < p_1 < 1$, exponential when $p_1 \in \{1, 1 - p_2\}$ and hypoexponential when $p_1 = 0$. When $p_2 \downarrow 0$, it becomes

$$G_1(t) = 1 - \mathrm{e}^{-\delta t} - (1 - p_1)\delta t \mathrm{e}^{-\delta t}, \quad t \geq 0. \tag{2}$$

We remind the reader that by *Coxian distribution* we mean the distribution of $\sum_{i=1}^{N} X_i$ where, for some $n \geq 1$, $N, X_1, \ldots, X_n$ are independent, N is a positive integer random variable with support $\{1, \ldots, n\}$ and $X_i \sim \exp(\lambda_i)$ for $1 \leq i \leq n$. It is a special case of a phase-type distribution.

It is interesting to observe (and elementary to verify) that for $0 < \beta < \delta$, the function $(1 - ce^{-\delta t} - (1 - c)e^{-\beta t})1_{[0,\infty)}(t)$ is a cdf (cumulative distribution function) if and only if it has the form on the right hand side of (1), with p_1, p_2 satisfying $\beta = (1 - p_2)\delta$ and $c = 1 - \frac{1-p_1}{p_2}$. In particular, necessarily,

$$1 \geq c = \frac{p_2 + p_1 - 1}{p_2} \geq \frac{p_2 - 1}{p_2} = -\frac{\beta}{\delta - \beta} \,. \tag{3}$$

The exponential case occurs when $c \in \{0, 1\}$, the hyperexponential case is when $0 < c < 1$, the hypoexponential case is when $c = -\frac{\beta}{\delta-\beta}$ and when $-\frac{\beta}{\delta-\beta} < c < 0$ it is the convolution of the $\exp(\delta)$ distribution with a mixture of an atom at zero and the $\exp(\beta)$ distribution.

Remark 1 Another blood bank application that gives rise to the jump size distribution in (2) is the case in which demands are for batches of size either one or two. Suppose that a proportion θ of the population (say, with weight below 40 kg) can be satisfied by one blood unit, whereas others need two units. Then (2) holds with $p_1 = \theta$.

Organization of the paper. Section 2 is devoted to an analysis of the $M/G/1 + D$ PIS. We obtain an integral equation for the density of V and present its solution, highlighting a particular decomposition property of both the workload in the $M/G/1 + D$ queue and the VOT in its PIS counterpart. Subsequently we use knowledge of $f(\cdot)$ to obtain the distribution of the time the shelf is empty, the outdating rate, the unsatisfied demand rate and the distribution of the number of items on the shelf. In Section 3 we consider a generalization of $M/G/1{+}D$ PIS: We assume that items are of two different quality levels, each with its own expiration date. Section 4 discusses the situation in which the quality of a blood item monotonously decreases with age. We close in Section 5 with a brief discussion of a dam model with limited accessibility whose structure is closely related to the model of Section 4.

Throughout the paper $x \wedge y = \min(x, y)$, $*$ denotes the convolution operator and we use the standard abbreviations a.s. for almost surely and PASTA for Poisson Arrivals See Time Averages.

2 The $M/G/1 + D$ PIS

In this section we consider the PIS with Poisson(μ) demand arrival process, and with the interarrival times of items being i.i.d. with distribution $G(\cdot)$, having a finite mean $1/\lambda$. The VOT process $\{V(t), t \geq 0\}$ is a regenerative process whose regeneration points are the moments of outdatings. It is related to the so-called $M/G/1 + D$ queue in which customers have a deterministic patience D in the sense that they join the system only if upon their arrival they find the workload level in the system (= their waiting time) below or equal to D and otherwise balk. The VOT process is in fact this $M/G/1 + D$ workload process where we remove all the idle times. Therefore, we will analyze this system first and then infer the results that we need for the VOT case.

We call the length of a period when the workload level is at zero an *idle* period. For the $M/G/1{+}D$ queue workload it is $\exp(\mu)$ distributed and for the $M/G/1 + D$ PIS VOT it is nonexistent. We call the length of a period when the process is above the level D an *emptiness* time. It is equal to the overshoot above the level D. For the PIS VOT, this is the time where there is no inventory and demands are therefore lost. That is, *emptiness* in this case means that the shelves are empty from inventory (even though the VOT is at a positive level). It is clear that the emptiness times have the same distribution for both the $M/G/1 + D$ queue workload process and the $M/G/1 + D$ PIS VOT process. Due to PASTA it will be easier to infer this distribution from the $M/G/1 + D$ queue workload (see Lemma 4), so that also for this reason it makes sense to treat that case first.

Although the more general $M/G/1{+}G$ queue has been considered numerous times in the literature (e.g., [4, 5, 9, 15, 24] and further references therein, there is a certain intriguing decomposition property that arises for the $M/G/1 + D$ which we have never seen before that makes it worthwhile to give the short analysis of this particular model from scratch, which also helps in making this paper more self contained at the price of a minimal cost in word count.

Let us assume throughout, without loss of generality, that $D = 1$. Then standard level crossing arguments and PASTA imply that if W has the workload distribution of the $M/G/1{+}1$ queue, then $F(x) = \left(p_0 + (1 - p_0)\int_0^x f(w)\mathrm{d}w\right) 1_{[0,\infty)}(x)$

and (a.s.)

$$(1-p_0)f(x)=\mu\int_{[0,x\wedge 1]}(1-G(x-w))F(\mathrm{d}w)=\rho\int_{[0,x]}g_e(x-w)F(\mathrm{d}w)\,, \quad (4)$$

where $\rho=\mu/\lambda$ and $g_e(x)=\lambda(1-G(x))$ is the stationary residual lifetime density associated with the distribution G (see also (9) of [15]). With $G_e(x)=\int_0^x g_e(w)\mathrm{d}w$ we therefore have by integration on both sides and interchanging the order of integration on the right, noting that $G_e(x-w)=0$ for $w>x$ (relevant for the case $x\le 1$), that

$$F(x)-p_0=\rho\int_{[0,1]}G_e(x-w)F(\mathrm{d}w)\,. \quad (5)$$

For $x\le 1$ we therefore have that $F(x)=p_0+\rho G_e*F(x)$, so that with

$$U(x)=\left(\sum_{n=0}^{\infty}\rho^n G_e^{*n}(x)\right)1_{[0,\infty)}(x)\,, \quad (6)$$

which is finite for any choice of ρ (e.g., [3, Sec. 4], noting that $G_e(0)=0$), we immediately have that $F(x)=p_0U(x)$ and thus, for all $x\ge 0$,

$$F(x)=p_0\left(1+\rho\int_{[0,1]}G_e(x-w)U(\mathrm{d}w)\right)\,. \quad (7)$$

Letting $x\to\infty$ implies that $1=p_0(1+\rho U(1))$, hence,

$$p_0=\frac{1}{1+\rho U(1)}\,, \quad (8)$$

from which it also follows that, for $x\ge 0$,

$$F(x)=p_0+(1-p_0)\int_{[0,1]}G_e(x-w)H(\mathrm{d}w)\,, \quad (9)$$

where

$$H(x)=\begin{cases}0, & x<0,\\ \frac{U(x)}{U(1)}, & 0\le x<1,\\ 1, & x\ge 1\,.\end{cases} \quad (10)$$

Remark 2 It is worthwhile noting the following.

1. Denoting $M(x)=U(x)-1=\sum_{n=1}^{\infty}\rho^n G_e^{*n}(x)$, and noting that $M(x)=\rho G_e*U(x)$, we have for $x\in[0,1]$ that the steady state distribution of the VOT is given by

$$P(V\le x)=P(W\le x|\,W>0)=\frac{F(x)-p_0}{1-p_0}=\frac{\rho^{-1}M(x)}{1+M(1)}. \quad (11)$$

2. If instead of the M/G/1+1 model we assume that whenever the process jumps over the level 1 then the excess (overshoot) is discarded, then repeating the analysis above, gives that here also $F(x) = p_0 U(x)$ on $[0, 1]$ and since $1 = F(1) = p_0 U(1)$, the steady state distribution of this (doubly reflected) process is H. This is also immediate from

$$P(W \le x | \, W \le 1) = \frac{F(x)}{F(1)} = \frac{p_0 U(x)}{p_0 U(1)} = H(x), \quad x \in [0, 1] \, . \tag{12}$$

Recalling that $M(x) = U(x) - 1$ and $U(0) = 1$, the cdf of the VOT version of this distribution (that is, discarding the idle times) is given by

$$\frac{H(x) - H(0)}{1 - H(0)} = \frac{U(x) - 1}{U(1) - 1} = \frac{M(x)}{M(1)} \, , \tag{13}$$

which, by (11), is also the same as $P(V \le x | \, V \le 1)$. For motivation of this version of the VOT process and its analysis when $G(\cdot)$ is exponential, see [6]. Also, see [15, 16] for different derivations of this proportionality result.

3. With $N_e(\cdot)$ denoting a renewal counting process with increment distribution $G_e(\cdot)$, then, since $P(N_e(x) \ge n) = G_e^{*n}(x)$ we have that

$$U(x) = \sum_{n=0}^{\infty} \rho^n P(N_e(x) \ge n) = E \sum_{n=0}^{N_e(x)} \rho^n = \begin{cases} \frac{E\rho^{N_e(x)+1} - 1}{\rho - 1}, & \rho \ne 1 \, , \\ 1 + EN_e(x), & \rho = 1 \, . \end{cases} \tag{14}$$

4. For a sanity check, when $G(x) = 1 - e^{-\lambda x}$, then $G_e(\cdot) = G(\cdot)$ and $N_e(x) \sim$ Poisson(λx). Thus, $EN_e(x) = \lambda x$ and, for $\lambda \ne \mu$,

$$E\rho^{N_e(x)} = e^{\lambda(\rho - 1)x} = e^{(\mu - \lambda)x}. \tag{15}$$

Therefore, in this case, together with (14), simple computations imply that

$$f(x) = \frac{\lambda}{U(1)} \, e^{\mu(x \wedge 1) - \lambda x} \, , \tag{16}$$

where it is interesting to note that when $\lambda = \mu$ then $f(x)$ is constant on $[0, 1]$. Also, as expected from the memoryless property of $G(\cdot)$, if V has density f, then the conditional distribution of $V - 1$ given that $V > 1$ is $\exp(\lambda)$, which is also the distribution of the overshoot above level 1.

The results here are consistent with (3.14) and (3.15) of [19], which have been established there via a different derivation.

5. When $\rho < 1$ then $P(W_{M/G/1} \le x) = (1 - \rho)U(x)$, where $W_{M/G/1}$ has the steady state workload (= waiting time) distribution in a standard stable M/G/1 queue. Therefore, in this case,

$$H(x) = P(W_{M/G/1} \le x | \, W_{M/G/1} \le 1) \, . \tag{17}$$

From the above, for any positive value of ρ, we have the following decomposition result.

Theorem 3 *With p_0 and H from (8) and (10), let $X_e \sim G_e$, $Z \sim H$ and $I \sim \text{Bernoulli}(1-p_0)$ be independent random variables. Then for $W \sim F$ we have that*

$$W \sim I(Z + X_e)\,. \tag{18}$$

Consequently, if V has the steady state distribution of the VOT process, then

$$V \sim Z + X_e\,, \tag{19}$$

and the distribution of V has a nonincreasing right continuous (version of a) density

$$f(x) = \int_{[0,1]} g_e(x-w)H(\mathrm{d}w) = \frac{\lambda}{U(1)}\int_{[0,x\wedge 1]} (1-G(x-w))U(\mathrm{d}w)\,, \tag{20}$$

(in particular, since $U(0)-U(0-)=1$ and $G(0)=0$, then $f(0)=\lambda/U(1)$), which satisfies the level crossing relation

$$f(x) = \mu\int_0^{x\wedge 1}(1-G(x-w))f(w)\mathrm{d}w + f(0)(1-G(x))\,. \tag{21}$$

We note that the relation (18) is similar to the relation between the steady state distribution of the workload in the G/G/1 queue and the steady state waiting time. See, e.g., p. 274 of [2] or p. 296 of [14].

Noting that, by PASTA, the tail of the steady state distribution of the overshoot above the level 1 is (cf. (4)):

$$\int_{[0,1]}(1-G(1+x-w))F(\mathrm{d}w) = (1-p_0)f(1+x)/\mu\,, \tag{22}$$

and since the tail of the steady state distribution of the emptiness time of the shelf in the corresponding $M/G/1+D$ PIS is the conditional (steady state) tail of the overshoot given that the jump exceeded the level one, we immediately have the following.

Corollary 4 *Let B have the steady state distribution of the emptiness time. Then*

$$P(B > x) = \frac{(1-p_0)f(1+x)/\mu}{(1-p_0)f(1)/\mu} = \frac{f(1+x)}{f(1)}\,. \tag{23}$$

Remark 5 Key performance measures for PIS are the rate of outdating λ^* and the rate of unsatisfied demand μ^*. Recalling (11), $f(0)=\lambda/U(1)$ (Theorem 3) and $U(1)=1+M(1)$, it immediately follows (PASTA) that

$$\mu^* = \mu P(V>1) = \mu\left(1-\frac{\lambda}{\mu}\frac{(U(1)-1)}{U(1)}\right) = \mu-\lambda+f(0)\,. \tag{24}$$

The conservation law $\lambda-\lambda^* = \mu-\mu^*$ (cf. [12]) implies that $\lambda^* = f(0)$, as is intuitively obvious: the outdating rate equals the rate of downcrossing level zero.

Remark 6 Let $N_V(\cdot)$ be a delayed renewal counting process with first increment distributed like V (Theorem 3) while the other increments have distribution $G(\cdot)$. Then the steady state distribution of the number of items K on the shelf is the distribution of $N_V(1)$. In particular,

$$P(K=0) = P(V>1) = \frac{\mu^*}{\mu} = \int_1^\infty f(x)\mathrm{d}x\,, \tag{25}$$

and, for $n \geq 1$,

$$P(K \geq n) = P(V + S_{n-1} \leq 1) = \int_0^1 G^{*(n-1)}(1-x)f(x)\mathrm{d}x\,, \tag{26}$$

where S_k are the partial sums of i.i.d. random variables with distribution $G(\cdot)$ (with $S_0 = 0$).

Example 7 Let us consider the solution of (21) for the special case of the item interarrival distribution $G(\cdot)$ given in (1). Either by differentiating (21) twice and some manipulations, or by taking Laplace transforms on both sides of (21), we can show that

$$f(x) = f(0)(A_1\mathrm{e}^{s_1x} + A_2\mathrm{e}^{s_2x}), \quad 0 \leq x \leq 1, \tag{27}$$

where s_1 and s_2 denote the plus and minus root of a quadratic equation:

$$s_{1,2} = \frac{1}{2}\left[\mu - (2-p_2)\delta \pm \sqrt{\Delta}\right], \tag{28}$$

with

$$\Delta = (\mu - (2-p_2)\delta)^2 + 4\mu\delta(2-p_1-p_2) - 4(1-p_2)\delta^2\,. \tag{29}$$

Furthermore,

$$A_1 = \frac{s_1 + \delta(2-p_1-p_2)}{s_1 - s_2}, \quad A_2 = \frac{s_2 + \delta(2-p_1-p_2)}{s_2 - s_1}. \tag{30}$$

Substituting this solution into (21) with $x \geq 1$, we find that

$$f(x) = C_1\mathrm{e}^{-\delta x} + C_2\mathrm{e}^{-(1-p_2)\delta x}, \quad x \geq 1, \tag{31}$$

where

$$C_1 = f(0)(1-\zeta)\left[1 + \sum_{i=1}^{2}\mu A_i \frac{\mathrm{e}^{\delta+s_i} - 1}{\delta + s_i}\right], \tag{32}$$

$$C_2 = f(0)\zeta\left[1 + \sum_{i=1}^{2}\mu A_i \frac{\mathrm{e}^{(1-p_2)\delta+s_i} - 1}{(1-p_2)\delta + s_i}\right]. \tag{33}$$

Finally, $f(0)$ readily follows via normalization: $\int_0^\infty f(x)\ \mathrm{d}x = 1$.

Alternatively, we could also obtain (27) from (11) by first computing $U(x)$, recalling that, for $x \in [0,1]$, $\int_0^x f(t)\mathrm{d}t = \frac{U(x)-1}{\rho U(1)}$ and finally differentiating. The idea is to note that with G given in (1) we have that $G_e(x)$ is also a linear combination of $e^{-\delta x}$ and $e^{-(1-p_2)\delta x}$ and therefore the LST (Laplace-Stieltjes transform) of G_e is a linear combination of $\frac{\delta}{\delta+\alpha}$ and $\frac{(1-p_2)\delta}{(1-p_2)\delta+\alpha}$ with the same coefficients. If we denote this LST by $\gamma_e(\alpha)$, then for $\alpha \geq 0$ such that $\rho\gamma_e(\alpha) < 1$ we have that the LST of U is

$$\int_{[0,\infty)} e^{-\alpha x} U(\mathrm{d}x) = \frac{1}{1 - \rho\gamma_e(\alpha)}\,, \tag{34}$$

where the right hand side can be written as a ratio of two quadratic functions. The roots of the denominator are precisely $s_{1,2}$ as defined in (28) and thus the right hand side is just 1 plus a linear combination of $\frac{1}{\alpha - s_1}$ and $\frac{1}{\alpha - s_2}$. Inversion gives a linear combination of $e^{s_1 x}$ and $e^{s_2 x}$ which, upon adding 1, gives $U(x)$. Recalling that $f(0) = \lambda/U(1)$, (27) can now be verified. The computation for $x > 1$ is the same as earlier. We prefer to skip the missing details.

3 An optional expiration date

In the models reviewed so far, there is the implicit assumption that all the items which arrive at the shelf are of equal quality. However, this assumption does not take into account that items may arrive at the shelf from different sources and therefore, their quality might be different. In the blood bank example it is natural to assume that blood doses whose source is from young donors are of higher quality and thus by regulation of longer maximum lifetime than blood doses from old donors. In this section we assume that there are two possible expiration dates: with probability p the expiration date is β, where the regulating constant $\beta \in (0, 1)$, and with probability $q = 1 - p$ the expiration date is 1, while the issuing policy is still FIFO.

To make this model a bit clearer recall Figure 1. Imagine that each of the dotted lines that connect the A_i's on the vertical and horizontal axes are assigned the tag *young* or *old* independently with probabilities q and $p = 1 - q$, respectively. All items begin their shelf lives when they first reach level one. Young items's shelf lives are equal to one and therefore their virtual outdate time (in the absence of demand) is when their dotted line hits the level zero. Old items shelf lives are equal to $\beta \in (0, 1)$ and therefore their virtual outdate time is when their dotted line hits the level $1 - \beta$ (β time units from their arrival time at level one). At this latter instance there are two possibilities. Either this old item is the most senior item and then there will be a jump up to the next dotted line, or it is not, which is a case that we will take into account and explain below.

Deriving the balance equation for the density $f(\cdot)$ of the VOT for this extension of the $M/G/1 + D$ PIS turns out to be a quite delicate exercise. We give this derivation below, resulting in Theorem 8. We also sketch how this balance equation can be solved.

For a given $G(\cdot)$ assume that $X_i \sim G(\cdot)$ are i.i.d., let $S_0 = 0$ and let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. $N(t) = \sup\{n | \, S_n \leq t\}$ is the associated renewal counting process. Let $N_q \sim \mathrm{Geo}(q)$ be an independent random variable.

If at a point of a jump the VOT is at level $w \geq 1-\beta$, then the jump size distribution of the VOT process is $G(\cdot)$.

On the other hand, if $0 \leq w < 1-\beta$ then the jump size of the VOT process is distributed like $S_{(N(1-\beta-w)+1)\wedge N_q}$, as we now show. If all of the items that arrive after the current one are of old-type, then the jump will be to the level of the first such item that did not expire yet. This level is $w + S_{N(1-\beta-w)+1}$ and the jump size is $S_{N(1-\beta-w)+1}$. Otherwise, the first young-type item arrives at the time point S_{N_q}. If $N_q < N(1-\beta-w)+1$, then the jump would be to the level $w + S_{N_q}$ and otherwise, it is the situation explained for the previous case. Therefore, the jump size should be distributed like

$$S_{N(1-\beta-w)+1} \wedge S_{N_q} = S_{(N(1-\beta-w)+1)\wedge N_q} \,. \tag{35}$$

We next determine its distribution. Denote $t = 1-\beta-w$.
(i) If $y \leq t$ then $S_{N(t)+1} > t \geq y$ and therefore $S_{(N(t)+1))\wedge N_q} > y$ if and only if $S_{N_q} > y$. In particular, for $w \leq x < 1-\beta$, there is an upcrossing of level x with probability $P(S_{N_q} > x-w)$. For the exponential case this is just $e^{-q\lambda(x-w)}$, which is consistent with [23].
(ii) If $y > t$ then $S_{N_q} \leq t \Leftrightarrow N(t) \geq N_q \Leftrightarrow N(t)+1 > N_q$. Therefore, we can write

$$\begin{aligned} S_{(N(t)+1)\wedge N_q} &= S_{N_q} 1_{\{N_q \leq N(t)\}} + S_{N(t)+1} 1_{\{N_q > N(t)\}} \\ &= S_{N_q} 1_{\{S_{N_q} \leq t\}} + S_{N(t)+1} 1_{\{N_q > N(t)\}} \,. \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} P(S_{(N(t)+1)\wedge N_q} > y) &= \underbrace{P(S_{N_q} \leq t, S_{N_q} > y)}_{0} + P(S_{N(t)+1} > y, N_q > N(t)) \\ &= Ep^{N(t)} 1_{\{S_{N(t)+1} > y\}} = Ep^{N(t)} 1_{\{N(y)-N(t)=0\}} \,, \end{aligned} \tag{37}$$

which, for $t = y$, becomes

$$P(S_{N_q} > y) = Ep^{N(y)} \,. \tag{38}$$

For the exponential case $N(\cdot)$ is a Poisson process and by the independence of $N(t)$ and $N(y)-N(t)$, the last expression becomes

$$Ep^{N(t)} P(N(y)-N(t)=0) = e^{-\lambda q t} \, e^{-\lambda(y-t)} \,. \tag{39}$$

In particular, for $x \geq 1-\beta$ and $w < 1-\beta$ we have that for the exponential case there is a jump above level x with probability

$$e^{-\lambda(q(1-\beta-w)+(x-(1-\beta)))} \,, \tag{40}$$

which, in this case as well, is consistent with [23].

To summarize, recalling (38), we have the following,

Theorem 8 *In the model of this section, the probability that, at a jump epoch, there is an upcrossing of level x from level w is given by*

$$U(w,x) = \begin{cases} P(S_{N_q} > x-w) = Ep^{N(x-w)}, & 0 \le w \le x < 1-\beta, \\ Ep^{N(1-\beta-w)} 1_{\{N(x-w)-N(1-\beta-w)=0\}}, & 0 \le w < 1-\beta \le x, \\ 1-G(x-w), & 1-\beta \le w \le x \wedge 1\,. \end{cases} \quad (41)$$

The balance equation for the VOT in the model of this section is (for the right continuous version of the density f)

$$f(x) = \mu \int_0^{x\wedge 1} U(w,x)f(w)\mathrm{d}w + pf(1-\beta)U(1-\beta,x)1_{(1-\beta,\infty)}(x) + f(0)U(0,x)\,. \quad (42)$$

Remark 9 We note the following.

1. For $0 \le w \le x = 1-\beta$ we have that
$$Ep^{N(1-\beta-w)} 1_{\{N(x-w)-N(1-\beta-w)=0\}} = Ep^{N(x-w)} \quad (43)$$
and for $1-\beta = w \le x$ we have that
$$Ep^{N(1-\beta-w)} 1_{\{N(x-w)-N(1-\beta-w)=0\}} = 1-G(x-w)\,. \quad (44)$$

2. We observe that $1-\beta$ is a point of discontinuity for $f(\cdot)$, since $f(1-\beta) - f((1-\beta)-) = pf(1-\beta)$ and thus $f((1-\beta)-) = qf(1-\beta)$. The intuitive argument behind this is the fact that $f(1-\beta)$ represents the rate of downcrossings at level $1-\beta$, a proportion q of which are not outdated.

3. With the notations of Remark 6, with $I_1, I_2, \ldots$ being i.i.d. Bernoulli(p) random variables, determining whether an item is young ($I_i = 0$) or old ($I_i = 1$), respectively, and with $J_n = \sum_{i=1}^n I_i$ ($J_0 = 0$), the number of young- and old-type items on the shelf is $N_V(1) - J_{N_V(1)}$ and $J_{N_V(1)} - J_{N_V(1-\beta)}$, respectively. Therefore, the total number of items on the shelf is $N_V(1) - J_{N_V(1-\beta)}$. When $G(\cdot)$ is the exponential distribution then, conditioned on V, one has to distinguish between the cases $V \le 1-\beta$, $1-\beta < V \le 1$ and $V > 1$. In particular, see also [23],
   (a) For $V \le 1-\beta$ the number of young and old items is distributed like $(1+X, Y)$ where X, Y are independent with $X \sim$ Poisson($q\lambda(1-V)$) and $Y \sim$ Poisson($p\lambda\beta$).
   (b) For $1-\beta < V \le 1$, they are distributed like $(1-I+X, I+Y)$, where X, Y, I are independent, where $I \sim$ Bernoulli(p), $X \sim$ Poisson($q\lambda(1-V)$) and $Y \sim$ Poisson($p\lambda(1-V)$).
   (c) For $V > 1$ both are zero.

4. The conservation law for this model version is
$$\mu - \mu^* = \lambda - f(0) - pf(1-\beta)\,. \quad (45)$$

Below we discuss how the balance equation (42) for the VOT can be formally solved; thereafter we briefly consider two examples.

(i) *The balance equation for* $x < 1 - \beta$. It follows from (41) that (42) now becomes:

$$f(x) = \mu \int_0^x P(S_{N_q} > x - w) f(w) \, \mathrm{d}w + f(0) P(S_{N_q} > x). \tag{46}$$

This is basically the form of (21) and we can use its solution (11) with $G(x)$ replaced by $\tilde{G}(x) = P(S_{N_q} \leq x)$ with $\tilde{\lambda} = q\lambda$, since

$$\frac{1}{\tilde{\lambda}} = ES_{N_q} = ES_1 \, EN_q = \frac{1}{\lambda} \frac{1}{q} \, . \tag{47}$$

We recall the LST of $U(\cdot)$ (for sufficiently large α) from (34), which holds for general $G(\cdot)$, not just for $G(\cdot)$ as given in (1), from which the LST of $M(\cdot)$ is obtained by subtracting one. Therefore the same can be done here, with $\tilde{\rho} = \mu/\tilde{\lambda}$ and $\tilde{G}$ replacing ρ and G. We note that the LST of $\tilde{G}$ is

$$Ee^{-\alpha S_{N_q}} = E\gamma(\alpha)^{N_q} = \frac{q\gamma(\alpha)}{1 - p\gamma(\alpha)} = \frac{q\lambda - q\alpha\gamma_e(\alpha)}{q\lambda + p\alpha\gamma_e(\alpha)} \tag{48}$$

where $\gamma(\alpha)$ is the LST of G and we recall that $\gamma_e(\alpha) = \alpha^{-1}\lambda(1 - \gamma(\alpha))$ is the LST of G_e. Therefore, the LST of $\tilde{G}_e$ is given by

$$\tilde{\gamma}_e(\alpha) = \frac{\tilde{\lambda}(1 - Ee^{-\alpha S_{N_q}})}{\alpha} = \frac{q\lambda\gamma_e(\alpha)}{q\lambda + p\alpha\gamma_e(\alpha)} \, . \tag{49}$$

For various $G(\cdot)$, including the exponential, hyperexponential, hypoexponential and Erlang distributions, one can analytically invert $(1 - \tilde{\rho}\tilde{\gamma}_e(\alpha))^{-1}$ and thus obtain $f(x)$ for $x \in [0, 1-\beta)$. One could also perform the inversion numerically, using a standard numerical inversion procedure; see, e.g., [1, 17].

(ii) *The balance equation for* $x \in [1 - \beta, 1]$. Now $U(w, x)$ does not have a pure convolution form. For particular choices of $G(\cdot)$ it is still possible to solve the resulting form of (42) analytically (see in particular Example 11 below), but in general it seems very difficult to obtain an explicit expression for $f(x)$ with $x \in [1-\beta, 1]$. However, formally the following approach is possible, which can be turned into a numerical procedure. Write (42) for $x \in [1 - \beta, 1]$ as the following Volterra integral equation of the second kind (cf. Chapter I of [22]):

$$f(x) = \mu \int_0^x U(w, x) f(w) \, \mathrm{d}w + Z(x), \tag{50}$$

where $Z(x)$ is known up to the unknown $f(0)$ – which has to be determined at the very end, via the normalizing condition $\int_0^\infty f(x)\mathrm{d}x = 1$. Now apply successive (so-called Picard) iteration to (50), yielding

$$f(x) = Z(x) + \sum_{n=1}^{\infty} \mu^n \int_0^x U^{*n}(w, x) Z(w) \, \mathrm{d}w, \tag{51}$$

where $U^{*1}(w,x) := U(w,x)$ and $U^{*n}(w,x) := \int_w^x U^{*(n-1)}(w,t)U(t,x)\ \mathrm{d}t$, $n = 2,3,\ldots$. If $Z(x)$ is absolutely integrable, then the infinite sum converges for all values of μ, cf. p. 16 of [22].

(iii) *The balance equation for* $1-\beta \le w \le x \wedge 1$. For $x \le 1$ we again have a pure convolution form for the yet unknown part $\int_{1-\beta}^x$ of the integral in (42); now see (i) above. For $x > 1$ the right hand side of that balance equation is known up to a constant, given that we then already know $f(x)$ for $x \le 1$.

Example 10 Let $G(x) = 1 - \mathrm{e}^{-\lambda x}$, i.e., items arrive according to a Poisson process with rate λ. The resulting PIS has already been studied in [23], with a slightly different motivation; there it is assumed that items which have not yet been taken after time β are inspected, and with probability $p = 1-q$ are found to be 'not in order', resulting in instantaneous removal from the shelf. As already indicated above, the kernel $U(w,x)$ in this case reduces to the kernel in [23] (where p and q are swapped). The resulting integral equation can be solved in fairly straightforward manner, leading to $f(x) = k_1\mathrm{e}^{(\mu-\lambda q)x}$ for $x \in (0, 1-\beta]$, $f(x) = k_2\mathrm{e}^{(\mu-\lambda)x}$ for $x \in (1-\beta, 1]$ and $f(x) = k_3\mathrm{e}^{-\lambda x}$ for $x > 1$.

Example 11 In Section 1, as a motivating example of $M/G/1{+}D$ PIS, we mentioned a distribution center assigning items alternately to two store locations. Under the assumption of items arriving at the distribution center according to a Poisson(λ) process, this results in Erlang($2,\lambda$) distributed interarrival times of items at a store location. Let us now (again) consider the balance equation (42) for $f(x)$.

(i) $x \in [0, 1-\beta)$. When $X_i \sim$ Erlang($2,\lambda$) it follows from (48) that

$$E\mathrm{e}^{-\alpha S_{N_q}} = \frac{(1-p)\lambda^2}{(\lambda+\alpha)^2 - p\lambda^2} = \frac{\lambda(1-\sqrt{p})}{\lambda(1-\sqrt{p})+\alpha}\,\frac{\lambda(1+\sqrt{p})}{\lambda(1+\sqrt{p})+\alpha}. \tag{52}$$

This reveals that S_{N_q} is distributed as the sum of two independent exponentially distributed random variables with parameters $\lambda(1-\sqrt{p})$ and $\lambda(1+\sqrt{p})$:

$$\begin{aligned} U(w,x) &= \frac{(1+\sqrt{p})e^{-\lambda(1-\sqrt{p})(x-w)} - (1-\sqrt{p})e^{-\lambda(1+\sqrt{p})(x-w)}}{2\sqrt{p}} \\ &= \left(\cosh(\sqrt{p}\,\lambda(x-w)) + \frac{1}{\sqrt{p}}\sinh(\sqrt{p}\,\lambda(x-w))\right)e^{-\lambda(x-w)}. \end{aligned} \tag{53}$$

It is not hard to verify that $f(x)$, in its turn, also is a mixture of two exponential terms.

(ii) $x \in [1-\beta, 1]$. We need to compute (37) for this Erlang case. Assume that $N_\lambda(t)$ is a Poisson process with rate λ. Then

$$\begin{aligned} Ep^{N(t)}1_{\{N(y)=N(t)=n\}} &= p^n P(N_\lambda(t) \ge 2n, N_\lambda(y) \le 2n+1) \\ &= p^n P(N_\lambda(t) = 2n)P(N_\lambda(y) - N_\lambda(t) \le 1) \\ &\quad + p^n P(N_\lambda(t) = 2n+1)P(N_\lambda(y) - N_\lambda(t) = 0) \\ &= p^n\left(\frac{(\lambda t)^{2n}}{(2n)!}(1+\lambda(y-t)) + \frac{(\lambda t)^{2n+1}}{(2n+1)!}\right)e^{-\lambda y}. \end{aligned} \tag{54}$$

Summation gives

$$Ep^{N(t)}1_{\{N(y)=N(t)\}} = \left(\cosh(\sqrt{p}\,\lambda t)(1+\lambda(y-t)) + \frac{1}{\sqrt{p}}\sinh(\sqrt{p}\,\lambda t)\right)e^{-\lambda y}\,. \tag{55}$$

Therefore the probability of an upcrossing of the level $x \geq 1-\beta$ starting from $w < 1-\beta$ is given by

$$\begin{aligned} U(w,x) = \Big(& \cosh(\sqrt{p}\,\lambda(1-\beta-w))(1+\lambda(x-(1-\beta))) \\ & + \frac{1}{\sqrt{p}}\sinh(\sqrt{p}\,\lambda(1-\beta-w))\Big)e^{-\lambda(x-w)}\,. \end{aligned} \tag{56}$$

We note that the right hand sides of (53) and (56) are equal for $x = 1-\beta$, as expected. It seems difficult to solve the integral equation with this $U(w,x)$.
(iii) $1-\beta \leq w \leq x \wedge 1$. The probability that there is an upcrossing of $x \geq w$ from level w is now given by

$$U(w,x) = 1 - G(x-w) = (1+\lambda(x-w))e^{-\lambda(x-w)}\,, \tag{57}$$

which, as expected, agrees with (56) when $w = 1-\beta$.

Therefore, the balance equation in this case is (42) where $U(w,x)$ is defined by (53), (56) and (57), for the cases $0 \leq w \leq x < 1-\beta$, $0 \leq w < 1-\beta \leq x$ and $1-\beta \leq w \leq x \wedge 1$, respectively.

4 Quality measured by age

In continuation of the argument that young items are considered to be of higher quality than old items, suppose that there exists a certain deterministic function $Q : [0,\infty) \to [0,\infty)$ that relates the remaining lifetime of an item to its quality. Obviously, the longer the remaining lifetime of an item, the higher is its quality, although this does not need to be assumed at the outset. Motivated by the blood bank example, the latter statement means that as the age of a blood portion increases (its remaining lifetime decreases), its quality decreases.

Assume that if $V(t)$ is the remaining lifetime of the oldest item on the shelf, then its measure of quality is $Q(V(t))$. When an item is put on the shelf, its quality level is $Q(1)$ since at this time its remaining lifetime is (without loss of generality) 1. When Q is nondecreasing, then the measure of quality that satisfied demands observe must be in $[Q(0), Q(1)]$ and quality levels which are higher than $Q(1)$ can only be observed at some embryonic state before their arrival at the shelf and are meaningless from the point of view of satisfied demands. Unsatisfied demands are obviously lost when they see a value which is higher than $Q(1)$.

If Q is sufficiently nice (Borel), then, since V is regenerative with finite mean regeneration epochs, the ergodic (almost sure long run average) distribution associated with the process $Q(V(t))$ is the distribution of $Q(V)$. For it to also have the limiting distribution it then suffices that Q is continuous.

Assume now that Q is continuous and strictly increasing, with $Q(0) = 0$ (perished items are worthless). Denote its inverse by $R(\cdot)$. Then $R(\cdot)$ is also continuous and strictly increasing. In addition, let us assume that $R(\cdot)$ is absolutely continuous with an (almost surely) finite and positive density denoted $R'(x)$ and let $r(x) = 1/R'(x)$.

Consider now (21). Taking $x = R(y)$ (the remaining lifetime of an item with quality level y), noting that $x \wedge 1 = R(y) \wedge R(Q(1)) = R(y \wedge Q(1))$ and that $0 = R(0)$, it becomes:

$$f(R(y)) = \mu \int_0^{R(y\wedge Q(1))} (1-G(R(y)-w))f(w)\mathrm{d}w + f(R(0))(1-G(R(y)))\,. \quad (58)$$

Making the change of variables $w = R(u)$, so that $\mathrm{d}w = \mathrm{d}u/r(u)$, and observing that the density f_q of $Q(V)$ is given by

$$f_q(y) = f(R(y))R'(y) = \frac{f(R(y))}{r(y)} = \frac{f(x)}{r(y)}\,, \quad (59)$$

we arrive at the level crossing equation

$$\begin{aligned} r(y)f_q(y) = \mu \int_0^{y\wedge Q^*} & (1 - G(R(y) - R(u)))f_q(u)\mathrm{d}u \\ & + r(0)f_q(0)(1 - G(R(y)))\,, \end{aligned} \quad (60)$$

with $Q^* = Q(1)$. Obviously, starting from this equation (with $Q^* = Q(1)$), reverse engineering leads back to (21).

Just like (21), also (60) is a level crossing balance equation. On the left, $r(y)$ is the downcrossing rate of the VOT-quality process when in state y, while on the right the two terms describe the rate of jumps that upcross y due to satisfied demands or outdatings. To see this, first observe that $R(u)$ is the VOT associated with quality u, so that $R(u) + X$ is the remaining time of the second oldest item, with quality $Q(R(u) + X)$, where $X \sim G$. Therefore, a jump (in quality) from u that upcrosses the level y occurs if and only if $Q(R(u)+X) > y$ which is equivalent to $X > R(y) - R(u)$.

The importance of the above structure is in allowing the modeling of an external controller that determines the local rate $r(y)$ at which the current broadcasted (by the controller) reading of quality level of the oldest item deteriorates.

Via Theorem 3, in particular (19), the above immediately implies the following for a VOT-quality process where quality deteriorates according to a (Borel and almost surely) positive and finite state-dependent rate function $r(\cdot)$ and demands are refused when Q exceeds some level $Q^* > 0$. With $R(x) = \int_0^x \frac{\mathrm{d}u}{r(u)}$ and $Q(t) = R^{-1}(t)$ we have that, if $Q(1) = Q^*$, then the steady state quality distribution is that of $Q(V) \sim Q(Z + X_e)$.

When $Q(1) \neq Q^*$ then, whenever $Q^* < \lim_{t\to\infty} Q(t)$ we take D such that $D = R(Q^*)$ (so that $Q(D) = Q^*$), let $r_D(x) = Dr(x)$, so that $R_D(x) = R(x)/D$

and $Q_D(t) = R_D^{-1}(t) = Q(Dt)$. This implies that in this case the steady state distribution is that of $Q_D(Z+X_e) = Q(D(Z+X_e))$. When $Q^* \ge \lim_{t\to\infty} Q(t)$, then all demands are satisfied.

To summarize:

Theorem 12 *Consider (60) (holding a.s.), where $r(\cdot)$ is a positive (Borel) function and $R(x) = \int_0^x \frac{du}{r(u)}$, assumed finite for all $x \in (0,\infty)$. Let $Q_\infty = \int_0^\infty \frac{du}{r(u)}$ (possibly infinite) and assume that $Q^* < Q_\infty$. Let $Q(\cdot)$ be the inverse of the function $R : [0,\infty) \to [0, Q_\infty)$ and let $D = R(Q^*)$. Then the density f_q is of a random variable distributed like $Q(D(Z+X_e))$, where Z, X_e are defined in Theorem 3.*

Therefore, whenever we can more explicitly compute the density f of V, as we have done in Section 2, and $Q^* < \lim_{t\to\infty} Q(t)$, then the steady state density of the VOT-quality process for a given rate function $r(\cdot)$ is that of $Q_D(V)$, namely

$$\frac{f(R_D(x))}{r_D(x)} = \frac{f(R(x)/D)}{Dr(x)} \,. \tag{61}$$

Computing $Eh(Q_D(V))$ for a sufficiently nice $h(\cdot)$, in particular moments, can then be carried out either directly with the density of $Q_D(V) = Q(DV)$ above or, when R can be easily inverted, via

$$Eh(Q_D(V)) = \int_0^\infty h(Q(Dt))f(t)\mathrm{d}t \,. \tag{62}$$

5 A related dam model with limited accessibility

A level crossing balance equation that is similar to (60) is

$$r(x)f(x) = \mu \int_0^{x\wedge 1} (1 - G(x-w))f(w)dw + r(0)f(0)(1 - G(x)) \,. \tag{63}$$

At first instance, one might (and we did) think that *this* balance equation, instead of (60), holds for the VOT density in case of quality measured by age. Actually, (63) holds for the buffer content in a dam model with compound Poisson input and state-dependent output rate $r(\cdot)$, where input is accepted only when the dam level is below one, and where also idle (empty dam) periods are removed. Accepted input may cause the level to increase beyond one, just like for the PIS VOT process.

With the experience from the previous section, (63) is equivalent to

$$\tilde f(t) = \mu \int_0^{t\wedge R(1)} (1 - G(Q(t) - Q(s)))\tilde f(s)ds + \tilde f(0)(1 - G(Q(t)) \,, \tag{64}$$

where $\tilde f(t) = r(Q(t))f(Q(t))$, where we recall that $Q(\cdot)$ is the inverse of $R(x) = \int_0^x \frac{\mathrm{d}u}{r(u)}$. This is in the same spirit as (60) and (21) relate to one another. Hence the solution of (63) immediately can be translated into a solution of (64). Special cases of (64) occur in several queueing, insurance and dam models; examples are Theorem 1 in [8] and Equation (15) in [11].

Below we very briefly discuss the solution of (63) for two cases; for a more extensive discussion of several cases we refer to [10].

Case 1: General positive $r(\cdot)$ and $\exp(\lambda)$ jump size distribution.
Differentiating both sides of (63) w.r.t. x readily results in $r(x)f'(x) = (\mu - \lambda r(x) - r'(x))f(x)$, and hence

$$f(x) = \frac{r(0)f(0)}{r(x)} \mathrm{e}^{-\lambda x + \int_0^x \frac{\mu}{r(w)} \mathrm{d}w}, \quad 0 \le x \le 1. \tag{65}$$

As expected, this solution is proportional to the solution for the unrestricted dam; cf. Equation (42) of [13]. For $x > 1$ we obviously have that $r(x)f(x) = Cf(0)\mathrm{e}^{-\lambda x}$ for some (now known) constant C; $f(0)$ follows from the normalizing condition $\int_0^\infty f(x)\mathrm{d}x = 1$.

Case 2: $r(x) = ax + b$ with $a, b > 0$, and $G(\cdot)$ as given in (1), with $p_2 > 0$.
Twice differentiating (63) w.r.t. x leads, after some manipulations, to the following second order ODE, where $r_1 := \frac{1-p_1}{p_2}$:

$$\begin{aligned} &(ax+b)f''(x) + [2a - \mu + (2-p_2)\delta(ax+b)]f'(x) \\ &+ [\mu(1-r_1)\delta + \mu r_1(1-p_2)\delta + (1-p_2)\delta^2(ax+b) + (2-p_2)\delta(a-\mu)]f(x) = 0. \end{aligned} \tag{66}$$

This ODE has the form $(A_2x+B_2)f''(x)+(A_1x+B_1)f'(x)+(A_0x+B_0)f(x) = 0$. It is item (equation) 2.1.2.108 on page 247 of [27]. Table 15 on that page presents its solution, distinguishing between four cases for A_0, A_1, A_2. Our example corresponds to the first case, where $A_2 \neq 0$, while $A_1^2 \neq 4A_0A_2$, leading to the following solution: with $k := \frac{\sqrt{A_1^2-4A_0A_2}-A_1}{2A_2} = (p_2-1)\delta$,

$$\begin{aligned} f(x) &= \mathrm{e}^{kx}\mathcal{J}\Big(\frac{B_2k^2+B_1k+B_0}{2A_2k+A_1}, \frac{A_2B_1-A_1B_2}{A_2^2}; \frac{-(x+\frac{B_2}{A_2})}{\frac{A_2}{2A_2k+A_1}}\Big) \\ &= \mathrm{e}^{-(1-p_2)\delta x}\mathcal{J}\Big(\frac{a-\mu r_1}{a}, \frac{2a-\mu}{a}; -p_2\delta(x+\frac{b}{a})\Big), \end{aligned} \tag{67}$$

where $\mathcal{J}(C_1, C_2; z)$ is a Bessel function of the first kind. Once more we have proportionality of $f(x)$ on $[0,1]$ to the solution of (63) for the *unrestricted* case where the integral runs from 0 to x for all $x > 0$. In the latter case one speaks of a shot-noise queue instead of a dam.

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